

COMPOSITION OPERATORS FROM LOGARITHMIC BLOCH SPACES TO WEIGHTED BLOCH SPACES

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ABSTRACT. We characterize the analytic self-maps ϕ of the unit disk \mathbb{D} in \mathbb{C} that induce continuous composition operators C_ϕ from the log-Bloch space $\mathcal{B}^{\log}(\mathbb{D})$ to μ -Bloch spaces $\mathcal{B}^\mu(\mathbb{D})$ in terms of the sequence of quotients of the μ -Bloch semi-norm of the n th power of ϕ and the log-Bloch semi-norm (norm) of the n th power F_n of the identity function on \mathbb{D} , where $\mu : \mathbb{D} \rightarrow (0, \infty)$ is continuous and bounded. We also obtain an expression that is equivalent to the essential norm of C_ϕ between these spaces, thus characterizing ϕ such that C_ϕ is compact. After finding a pairwise norm equivalent family of log-Bloch type spaces that are defined on the unit ball \mathbb{B}_n of \mathbb{C}^n and include the log-Bloch space, we obtain an extension of our boundedness/compactness/essential norm results for C_ϕ acting on \mathcal{B}^{\log} to the case when C_ϕ acts on these more general log-Bloch-type spaces.

Keywords: Bloch spaces, Composition operators.

MSC 2010: 30D45, 32A30, 47B33.

1. INTRODUCTION

1.1. Domains considered and weighted Bloch spaces. Let \mathbb{D} denote the unit disk in the complex plane \mathbb{C} , and denote by $\mathcal{H}(\mathbb{D})$ the linear space of all holomorphic functions on \mathbb{D} . Throughout this paper, \log denotes the natural logarithm function, and μ denotes what we call a *weight* on \mathbb{D} ; that is, μ is a bounded, continuous and strictly positive function defined on \mathbb{D} . The μ -Bloch space $\mathcal{B}^\mu(\mathbb{D})$, which we denote more briefly by \mathcal{B}^μ , consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_\mu := \sup_{z \in \mathbb{D}} \mu(z) |f'(z)| < \infty.$$

μ -Bloch spaces are called *weighted Bloch spaces*. When $\mu(z) = 1 - |z|^2$, \mathcal{B}^μ becomes the classical Bloch space $\mathcal{B}(\mathbb{D})$. If $\alpha \geq 0$ and $\mu : \mathbb{D} \rightarrow (0, 1)$ is given by $\mu(z) = (1 - |z|^2)^\alpha$, then we denote $\|\cdot\|_\mu$ by $\|\cdot\|_\alpha$, and in this case, $\mathcal{B}^\mu(\mathbb{D})$ is denoted by $\mathcal{B}^\alpha(\mathbb{D})$, the so-called α -Bloch space of \mathbb{D} . For weights μ on \mathbb{D} , a Banach space structure (cf. [25]) on $\mathcal{B}^\mu(\mathbb{D})$ arises if it is given the norm

$$\|f\|_{\mathcal{B}^\mu} := |f(0)| + \|f\|_\mu.$$

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The second author is partially supported by a Research Sabbatical at Fullerton College, and his research interaction with students is partially supported by ENGAGE in STEM (Encouraging New Graduates and Gaining Expertise in Science, Technology, Engineering and Math), a Special Program funded by the United States Department of Education Hispanic Serving Institutions (HSI) STEM and Articulation Programs cooperative arrangement grant project involving California State University, Fullerton, Fullerton College, and Santa Ana College.

The fourth author has been partially supported by the Comisión de Investigación, Universidad de Oriente, Project CI-2-010301-1678-10.

These Banach spaces provide a natural setting in which one can study properties of various operators. For instance, K. Attele in [1] proved that if $\mu_1(z) := w(z) \log \frac{2}{w(z)}$, where $w(z) := 1 - |z|^2$ and $z \in \mathbb{D}$, then the Hankel operator H_f induced by a function f in the Bergman space $A^2(\mathbb{D})$ (see [8, Ch. 2]) is bounded if and only if $f \in B^{\mu_1}(\mathbb{D})$, thus giving one reason, and not the only reason, why log-Bloch-type spaces are of interest.

1.2. Definition of the log-Bloch space. For notational convenience, we will state and prove our main results for composition operators acting on the μ -Bloch space $\mathcal{B}^{v_{\log}}(\mathbb{D})$, where $v_{\log} : \mathbb{D} \rightarrow (0, \infty)$ is the weight given by

$$(1) \quad v_{\log}(z) = (1 - |z|) \log \left(\frac{3}{1 - |z|} \right).$$

We will also denote $\mathcal{B}^{v_{\log}}(\mathbb{D})$ by \mathcal{B}^{\log} and the semi-norm $\|f\|_{\mathcal{B}^{\log}} - |f(0)|$ by $\|f\|_{\log}$ rather than $\|f\|_{v_{\log}}$.

1.3. Definition of a composition operator. During the past decade, there has been a surge in new results concerning various linear operators $L : X \rightarrow Y$ where at least one of the spaces X and Y is a space of functions satisfying a Bloch-type growth condition. A steadily increasing amount of attention has been paid to the case when $L = C_{\phi}$, a so-called *composition operator*, which we now define.

Let \mathcal{H}_1 and \mathcal{H}_2 be two linear subspaces of $\mathcal{H}(\mathbb{D})$. If ϕ is a holomorphic self-map of \mathbb{D} , such that $f \circ \phi$ belongs to \mathcal{H}_2 for all $f \in \mathcal{H}_1$, then ϕ induces a linear operator $C_{\phi} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ defined by

$$C_{\phi}(f) := f \circ \phi.$$

C_{ϕ} is called the *composition operator* with *symbol* ϕ . Composition operators continue to be widely studied on various subspaces of $\mathcal{H}(\mathbb{D})$. A standard introductory reference for the theory of composition operators is the monograph by C. Cowen and B. MacCluer [8], and another useful introduction to composition and other operators, particularly on Bloch-type spaces, is contained in the book by K. Zhu [34]. A lively introduction to composition operators on analytic function spaces of one complex variable is given in J. Shapiro's book on the subject [19]. We care here about C_{ϕ} as it acts between $X = \mathcal{B}^{\log}$ and $Y = \mathcal{B}^{\mu}$ for an arbitrary weight μ on \mathbb{D} , and later in the paper, we will consider C_{ϕ} as it acts on a more general family of spaces that include \mathcal{B}^{\log} .

It is natural to consider extensions of the above results, and the results of the present paper, to a generalization of Bloch-type spaces, called Bloch-Orlicz spaces, the first of which was introduced in [17]: Let μ be a weight on \mathbb{D} , and let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing, convex function such that $\Phi(0) = 0$, with $\lim_{t \rightarrow \infty} \Phi(t) = \infty$. Then we define the (μ, Φ) -Bloch-Orlicz space $\mathcal{B}_{\Phi}^{\mu}(\mathbb{D})$ to be the collection of all $f \in \mathcal{H}(\mathbb{D})$ such that there is a $\lambda > 0$ with

$$\sup_{z \in \mathbb{D}} \mu(z) \Phi(\lambda |f'(z)|) < \infty.$$

Note that if $\mu(z) := 1 - |z|^2$ above, then J. Ramos Fernandez' "Bloch-Orlicz space" is obtained as he defined it in [17], and upon which he studied superposition operators jointly with R. Castillo and M. Salazar in [3]. The theory of composition operators from and/or to \mathcal{B}_{Φ}^{μ} , which could be developed alongside the parallel, emerging theory of composition operators from and/or to "Orlicz-ised" transformations of

classical integrally defined function spaces, say, the Hardy-Orlicz (cf. [18]) and Bergman-Orlicz (cf. [20]) spaces, seems interesting.

We also point out here that log Bloch-type spaces are not simply “made-up spaces.” Showing that outer functions on \mathcal{B} are weak*-cyclic, L. Brown and A. L. Shields proved an auxiliary result in [2] which says that when $k = 2$ and $\theta = 1$, the space $\mathcal{B}_{k,\theta}^{\log}$ that we define in the last section of this paper coincides with the multiplier space of \mathcal{B} . In turn, it seems rather fundamental and natural to study the size of composition from log-Bloch type spaces to other (more general) weighted Bloch spaces.

1.4. Some related results concerning composition operators on Bloch-type spaces. In [14], K. Madigan and A. Matheson characterized the maps ϕ that generate, respectively, continuous and compact composition operators C_ϕ on \mathcal{B} . In turn, their results were extended by Xiao [29] to the α -Bloch spaces $\mathcal{B}^\alpha(\mathbb{D})$ for $\alpha > 0$ and by Yoneda [31] to \mathcal{B}^{\log} .

After he introduced a more general family of log-Bloch-type spaces that include \mathcal{B}^{\log} in [22], S. Stević obtained, jointly with R. Agarwal in [25], function theoretic characterizations of holomorphic functions ψ and holomorphic self-maps ϕ of \mathbb{D} such that the weighted composition operator $W_{\psi,\phi}$ on these spaces defined by $W_{\psi,\phi}(f) = \psi(f \circ \phi)$ is bounded or compact from these spaces to \mathcal{B}^μ where μ is a weight. Also, in [32], X. Zhang and J. Xiao characterized the holomorphic functions ψ on the complex Euclidean unit ball \mathbb{B}_n of \mathbb{C}^n , together with the holomorphic self-maps ϕ of \mathbb{B}_n , such that $W_{\psi,\phi}$ is bounded or compact between similarly defined μ -Bloch spaces of \mathbb{B}_n . For $n > 1$, it is required here that μ be a so-called *normal* function. The results of Zhang and Xiao were extended by H. Chen and P. Gauthier [4] to the μ -Bloch spaces of \mathbb{B}_n for which μ is a positive and non-decreasing continuous function such that $\mu(t) \rightarrow 0$ as $t \rightarrow 0$ and $\mu(t)/t^\delta$ is decreasing for small t and for some $\delta > 0$.

Other compactness criteria for composition operators on Bloch spaces have been found by M. Tjani [26], and more recently, H. Wulan, D. Zheng, and K. Zhu [28] proved the following result:

Theorem 1.1. [28] *Let ϕ be an analytic self-map of \mathbb{D} . Then C_ϕ is compact on \mathcal{B} if and only if*

$$\lim_{j \rightarrow \infty} \|\phi^j\|_{\mathcal{B}} = 0.$$

In [10], J. Giménez, R. Malavé, and J. C. Ramos Fernández extended the results of [14] to certain μ -Bloch spaces, where the weight μ can be extended to a non-vanishing, complex-valued holomorphic function that satisfies a reasonable geometric condition on the Euclidean disk $D(1, 1)$. Ramos Fernández in [17] introduced Bloch-Orlicz spaces, to which he extended all of the results mentioned above.

1.5. The essential norm of an operator. The essential norm of a continuous linear operator T between normed linear spaces X and Y is its distance from the compact operators; that is, $\|T\|_e^{X \rightarrow Y} = \inf \{\|T - K\| : K \text{ is compact}\}$, where $\|\cdot\|$ denotes the operator norm. Notice that $\|T\|_e^{X \rightarrow Y} = 0$ if and only if T is compact, so that estimates on $\|T\|_e^{X \rightarrow Y}$ lead to conditions for T to be compact.

1.6. Previous results on the essential norm of C_ϕ on Bloch-type spaces.

The essential norm of a composition operator on $\mathcal{B}(\mathbb{D})$ was calculated by A. Montes Rodríguez in [15]. He obtained similar results for essential norms of weighted composition operators between weighted Banach spaces of analytic functions in [16]. Other results in this direction appear in the paper by M. D. Contreras and A. G. Hernández Díaz in [5]; in particular, formulas for the essential norm of weighted composition operators on the α -Bloch spaces of \mathbb{B}_n were obtained (see also the paper of B. MacCluer and R. Zhao [13]). Recently, many extensions of the above results have appeared in the literature; for instance, the reader is referred to the paper of R. Yang and Z. Zhou [30] and several references therein. Zhao in [33] gave a formula for the essential norm of $C_\phi : \mathcal{B}^\alpha(\mathbb{D}) \rightarrow \mathcal{B}^\beta(\mathbb{D})$ in terms of an expression involving norms of powers of ϕ . More precisely, he showed that

$$\|C_\phi\|_e^{\mathcal{B}^\alpha(\mathbb{D}) \rightarrow \mathcal{B}^\beta(\mathbb{D})} = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{j \rightarrow \infty} j^{\alpha-1} \|\phi^j\|_{\mathcal{B}^\beta(\mathbb{D})}.$$

It follows from the discussion at the beginning of this paragraph that $C_\phi : \mathcal{B}^\alpha(\mathbb{D}) \rightarrow \mathcal{B}^\beta(\mathbb{D})$ is compact if and only if

$$\lim_{j \rightarrow \infty} j^{\alpha-1} \|\phi^j\|_{\mathcal{B}^\beta(\mathbb{D})} = 0.$$

O. Hyvärinen, M. Kemppainen, M. Lindström, A. Rautio, and E. Saukko in [11] recently obtained necessary and sufficient conditions for boundedness and an expression characterizing the essential norm of a weighted composition operator between general weighted Bloch spaces \mathcal{B}^μ , under the technical requirements that μ is radial, and that it is non-increasing and tends to zero toward the boundary of \mathbb{D} . The results in the present paper, in contrast, examine a more concrete domain space for C_ϕ but now for the case when the target space of C_ϕ is *any* weighted Bloch type space not necessarily having the aforementioned requirements.

1.7. Objectives and organization of the paper. The goal of this paper is to extend the results in [28] for composition operators on $\mathcal{B}(\mathbb{D})$ and [33] for composition operators between α -Bloch spaces of \mathbb{D} to the more general case of composition operators between log Bloch-type spaces (including the so-called log Bloch space as a special case) and μ -Bloch spaces. We will first show that if μ is a weight on \mathbb{D} and $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic, then the following statements hold:

- (Theorem 3.1 below) $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$ is bounded if and only if

$$\sup_{j \in \mathbb{N}} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} < \infty.$$

- (Theorem 5.4 below) $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$ is compact if and only if

$$\lim_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} = 0.$$

- (Theorem 5.1 below) If $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$, then

$$\|C_\phi\|_e^{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} \sim \limsup_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}}.$$

Above, and in what follows, $(F_j)_{j \in \mathbb{N}}$ denotes the sequence of monomial functions on \mathbb{D} given by $F_j(z) = z^j$, and, for two positive variable quantities A and B , we write $A \sim B$ and say that A is *equivalent to* B if and only if there is a positive constant K such that $\frac{1}{K}A \leq B \leq KA$ for all values of A and B .

In the section that follows, we will obtain results that are auxiliary to proving Theorem 3.1, which we state and prove in Section 3. After stating and proving some auxiliary compactness results in Section 4, we prove Theorems 5.1 and 5.4 in Section 5. In Section 6, we will point out that \mathcal{B}^{\log} , which has been called the “log-Bloch space”, is a member of a two-parameter, pairwise norm-equivalent family of log-Bloch-type spaces that we call (k, θ) -log Bloch spaces, this particular log-Bloch space corresponding to $k = 1$ and $\theta = 3$. In Section 7, we will immediately thus obtain that all of our results for composition operators hold more generally when C_ϕ acts on the (k, θ) -log Bloch spaces for $k = 1$ or 2 and $\theta \geq e$.

2. AUXILIARY FACTS

Throughout the rest of this paper, we let $L = 1 - 1/\log 3$. The following lemma, which we prove for the sake of completeness, will only be needed to prove the one that follows it:

Lemma 2.1. *Define $A : [1, \infty) \rightarrow (0, 1]$ by*

$$A(x) = \left(\frac{x}{x+L} \right)^{x-1}.$$

Then we have that

$$\inf_{x \geq 1} A(x) = \lim_{x \rightarrow \infty} A(x) = e^{-L}.$$

Proof: We have that

$$\lim_{x \rightarrow \infty} \log[A(x)] = \lim_{x \rightarrow \infty} -L \frac{x-1}{x+L} \log \left(1 + \frac{-L}{x+L} \right)^{\frac{x+L}{-L}} = -L.$$

The second equation in the statement of the lemma immediately follows. The first equation is obtained by observing that

$$\frac{d}{dx}[\log A(x)] = \log \frac{x}{x+L} + \frac{Lx-L}{x(x+L)} < \log \frac{x}{x+L} + \frac{L}{x+L} = \log \frac{x}{x+L} - \frac{x}{x+L} + 1,$$

which is negative, since $\log \eta - \eta + 1$ takes on values $-\infty$ and 0 at $\eta = 0$ and $\eta = 1$ respectively and is strictly increasing in $\eta \in (0, 1)$. It follows that the expression $\log A(x)$ (and, in turn, $A(x)$) is strictly decreasing on $[1, \infty)$. The first equation in the statement of the lemma then follows from this fact and the second equation in the statement of the lemma. \blacksquare

In what follows, we will have need for a sequence $(r_j)_{j \in \mathbb{W}}$, where \mathbb{W} denotes the whole numbers. We define

$$(2) \quad r_0 = 0 \quad \text{and} \quad r_j = 1 - \frac{L}{j+L} \quad \text{for each } j \in \mathbb{N}.$$

The sequence $(r_j)_{j \in \mathbb{W}}$, lies in $[0, 1)$, is strictly increasing and satisfies $r_j \rightarrow 1^-$ as $j \rightarrow \infty$. If $\phi : \mathbb{D} \rightarrow \mathbb{D}$ and $j \in \mathbb{N}$, then we define

$$A_j^\phi := \{z \in \mathbb{D} : r_{j-1} \leq |\phi(z)| < r_j\}.$$

Also needed in the proofs of both of our results on composition operators is the following sequence of functions $(h_j)_{j \in \mathbb{N}} : (0, 1) \rightarrow (0, \infty)$ given by

$$h_j(t) = \frac{j}{\log(j+1)} t^{j-1} (1-t) \log \left(\frac{3}{1-t} \right),$$

and the following fact about these functions:

Lemma 2.2. *Let $j \in \mathbb{N}$. Then h_j is decreasing on $[r_{j-1}, 1)$. Also, we have that*

$$(3) \quad h_j(t) \geq \frac{L}{2e^L} \quad \text{for all } t \in [r_{j-1}, r_j].$$

Proof. It suffices to show that h_j is decreasing on $[r_{j-1}, r_j]$ and bounded below by the right side of the above inequality on the same interval, for each $j \in \mathbb{N}$. Let $j \in \mathbb{N}$. Differentiating, we obtain that for all $t \in (0, 1)$,

$$h'_j(t) = \frac{j}{\log(j+1)} t^{j-2} \left[(j-1-jt) \log \left(\frac{3}{1-t} \right) + t \right].$$

Since $L \in (0, 1)$, we have that for all $t \in [r_{j-1}, 1)$ the inequality $j-1-jt < 0$ holds. This fact, the above formula for $h'_j(t)$, and the fact that $\log \left(\frac{3}{1-t} \right) \geq \log(3)$ holds for all $t \in (0, 1)$, together imply that for all $t \in [r_{j-1}, 1)$,

$$h'_j(t) \leq \frac{j}{\log(j+1)} t^{j-2} [(j-1-jt) \log(3) + t] < 0,$$

which in turn implies that h_j is decreasing on $[r_{j-1}, 1)$. The first statement in the lemma then immediately follows from this statement, and the fact that $(r_j)_{j \in \mathbb{N}}$ is increasing toward 1 with values in $[0, 1)$.

By the first statement in the lemma, we have that for any $j \in \mathbb{N}$ and all $t \in [r_{j-1}, r_j]$,

$$\begin{aligned} h_j(t) &\geq h_j(r_j) \\ &= \frac{\log \left[\frac{3}{L}(j+L) \right]}{\log(j+1)} \left[\frac{j}{j+L} \right]^{j-1} \left[\frac{jL}{j+L} \right] \\ &> 1 \left(\frac{j}{j+L} \right)^{j-1} \frac{L}{2} \\ &\geq e^{-L} \frac{L}{2} = \frac{L}{2e^L}. \end{aligned}$$

The inequality above is due to Lemma 2.1. The lemma's statement (3) follows. ■

The corollary that proceeds the following lemma, and not the lemma itself, will be used throughout this paper; nevertheless, the lemma and its proof may be of interest. These results require defining, for all $j \in \mathbb{N}$, $H_j : (0, 1) \rightarrow \mathbb{R}$, by

$$H_j(t) = \log(j+1)h_j(t).$$

To avoid the appearance of complex fractions in equations involving limits in the statements of some of our results, here and throughout the rest of this paper, we say that a real sequence $(a_j)_{j \in \mathbb{N}}$ is *asymptotic to* another real sequence of non-zero real numbers $(b_j)_{j \in \mathbb{N}}$ and write " $a_j \simeq b_n$ as $j \rightarrow \infty$ " if and only if

$$\lim_{j \rightarrow \infty} \frac{a_j}{b_j} = 1.$$

Lemma 2.3. *The following statements hold:*

(A) *For all such j with $j \geq 11$, there is a unique $t_j \in (0, 1)$ such that $H_j(t_j)$ is the absolute maximum of H_j .*

(B) *The sequence $(t_j)_{j \in \{11, 12, 13, \dots\}}$ satisfies the following three relations:*

$$(4) \quad \lim_{j \rightarrow \infty} t_j = 1^-,$$

where “ $-$ ” here denotes that t_j tends to 1 from the left,

$$(5) \quad t_j \simeq 1 - \frac{1}{j} + \frac{1}{j \log(3j)} \text{ as } j \rightarrow \infty, \text{ and}$$

$$(6) \quad \lim_{j \rightarrow \infty} [j(1 - t_j)] = 1.$$

(C) *We have that*

$$\max_{0 < t < 1} H_j(t) \simeq \frac{1}{e} \log(j+1), \text{ as } j \rightarrow \infty.$$

Proof. (A) Let $j \in \{11, 12, 13, \dots\}$. We define $g_j : (0, 1) \rightarrow \mathbb{R}$ by

$$g_j(t) = \{(j-1) - jt\} \log\left(\frac{3}{1-t}\right) + t,$$

which implies the following estimates, the strict inequality below being due to the assumption that $j \geq 11$:

$$g'_j(t) = j + 1 - \frac{1}{1-t} - j \log\left(\frac{3}{1-t}\right) \leq j + 1 - \log(3)j < 0.$$

Therefore, g_j is strictly decreasing, and since we also have that

$$\lim_{t \rightarrow 0^+} g_j(t) = (j-1) \log(3) > 0 \text{ and } \lim_{t \rightarrow 1^-} g_j(t) = -\infty,$$

there is a unique $t_j \in (0, 1)$ such that

$$(7) \quad g_j(t_j) = 0.$$

These facts establish that $g_j(t) > 0$ whenever $t \in (0, t_j)$ and that $g_j(t) < 0$ whenever $t \in (t_j, 1)$. This statement, and a direct calculation that

$$H'_j(t) = jt^{j-2}g_j(t) \text{ for all } t \in (0, 1),$$

together imply that H_j has a unique absolute maximum at t_j , as claimed.

(B) Again, let $j \in \{11, 12, 13, \dots\}$. Subsequent addition of t_j and division by j^2 on both sides of Equation (7) gives that

$$(8) \quad \left(\frac{j-1}{j} - t_j\right) \log\left(\frac{3}{1-t_j}\right) = -\frac{t_j}{j}.$$

Since $t_j \in (0, 1)$ for all $j \in \{11, 12, 13, \dots\}$, the right side of Equation (8) above has limit 0 as $j \rightarrow \infty$, and since the second factor of the left side of this equation is bounded below by $\log 3$, we deduce that

$$\lim_{j \rightarrow \infty} \left(t_j - \frac{j-1}{j}\right) = 0.$$

Equation (4) then follows from the triangle inequality, and in turn, subsequent applications of Equation (4), and the fact that the right side of Relation (5) tends to 1 as $j \rightarrow \infty$, verify that Relation (5) holds.

To prove Equation (6), which we will use to prove Part (C), we employ a technique that is inspired by methods for solving singularly perturbed non-linear equations (cf. [21]) as follows: We again use the fact that the second factor in the left side of Equation (8) is bounded below by $\log 3$ to obtain that

$$\frac{j-1}{j} - t_j = -\frac{t_j}{j \log \frac{3}{1-t_j}},$$

which is equivalent to the relation

$$j-1-jt_j = -\frac{t_j}{\log \frac{3}{1-t_j}}.$$

It follows that

$$j-jt_j = 1 - \frac{t_j}{\log 3 - \ln(1-t_j)}.$$

By Equation (4), the right side of the above equation has limit 1 as $j \rightarrow \infty$, and Equation (6) is now verified. Thus the proof of Part (B) of the lemma is now complete.

(C) We first point out that

$$j \log t_j = j \frac{\log(1+[t_j-1])}{t_j-1} (t_j-1) = j(t_j-1) \frac{\log(1+[t_j-1])}{t_j-1}.$$

By Equation (6) and a combination of Equation (4) and a L'Hopital's rule manipulation of the fractional expression above, it follows that the left side of the above equation must tend to $-1(1) = -1$. It follows from this fact and another application of Equation (4) that $(j-1) \log t_j \rightarrow -1$ as $j \rightarrow \infty$. Therefore, we have that

$$(9) \quad \lim_{j \rightarrow \infty} t_j^{j-1} = \frac{1}{e}.$$

Furthermore, Equation (6) also allows us to observe that

$$(10) \quad \frac{\log\left(\frac{3}{1-t_j}\right)}{\log j} = \frac{\log(3j)}{\log j} - \frac{\log(j(1-t_j))}{\log j} \rightarrow 1 \text{ as } j \rightarrow \infty.$$

Part (C) of the lemma then follows from the following chain of equations, the first of Equations (11) below following from, respectively, Equations (10), (6), and (9):

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{e}{\log j} \max_{0 < t < 1} H_j(t) &= \lim_{j \rightarrow \infty} \frac{e}{\log j} (1-t_j) \log\left(\frac{3}{1-t_j}\right) t_j^{j-1} \\ &= e \lim_{j \rightarrow \infty} \frac{\log\left(\frac{3}{1-t_j}\right)}{\log j} j(1-t_j) t_j^{j-1} \\ &= e(1)(1) \frac{1}{e} = 1. \end{aligned} \quad \blacksquare$$

We will have use for the following immediate consequence of Lemma 2.3:

Corollary 2.4.

$$\|F_j\|_{\log} \simeq \frac{\log(j+1)}{e} \text{ as } j \rightarrow \infty.$$

3. CONTINUITY OF COMPOSITION OPERATORS FROM \mathcal{B}^{\log} TO \mathcal{B}^μ

In this section, we obtain the following norm growth-rate characterization of the holomorphic self-maps ϕ of \mathbb{D} for which $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$ is continuous, where μ is a fixed weight on \mathbb{D} :

Theorem 3.1. *Suppose that $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$. Then C_ϕ is bounded if and only if*

$$(11) \quad \sup_{j \in \mathbb{N}} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} < \infty.$$

Proof. \Rightarrow : Suppose first that $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$ and that C_ϕ is bounded. Then there is an $M \geq 0$ such that $\|C_\phi f\|_{\mathcal{B}^\mu} \leq M\|f\|_{\mathcal{B}^{\log}}$ for all functions $f \in \mathcal{B}^{\log}$. This statement and the facts that $C_\phi(F_j) = \phi^j$, $F_j \in \mathcal{B}^{\log}$ and $F_j(0) = 0$ for all $j \in \mathbb{N}$ together imply that for all of these j 's, we have

$$\frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} = \frac{\|C_\phi(F_j)\|_\mu}{\|F_j\|_{\log}} \leq \frac{\|C_\phi(F_j)\|_{\mathcal{B}^\mu}}{\|F_j\|_{\log}} \leq \frac{M\|F_j\|_{\mathcal{B}^{\log}}}{\|F_j\|_{\log}} = \frac{M\|F_j\|_{\log}}{\|F_j\|_{\log}} = M.$$

Inequality (11) immediately follows.

\Leftarrow : Suppose now that Inequality (11) holds. To show that C_ϕ is bounded, we first show that there is an $\tilde{L} > 0$ such that for all $f \in \mathcal{B}^{\log}$,

$$(12) \quad \|f \circ \phi\|_\mu \leq \tilde{L}\|f\|_{\log}.$$

To prove this statement, let $z \in \mathbb{D}$ be fixed. Then there is a $j \in \mathbb{N}$ such that $|\phi(z)| \in A_j^\phi$. Note that j here depends on ϕ and z . There are two cases to consider: either $j = 1$ or $j \geq 2$. Suppose first that $j = 1$, so that in particular, $|\phi(z)| \leq r_1$. Since $t \log(3/t)$ is increasing and positive in t and we have now that $1 - r_1 \leq 1 - |\phi(z)|$,

$$v_{\log}[\phi(z)] \geq (1 - r_1) \log \frac{3}{1 - r_1} = \frac{L}{1 + L} \log \frac{3}{\frac{L}{1+L}} = \frac{L}{1 + L} \log \frac{3(1+L)}{L} > 0.$$

Note that the rightmost nonzero quantity above, which we now denote more briefly by L_1 , is a constant that depends neither on z nor ϕ . It follows that

$$(13) \quad \begin{aligned} \mu(z) |f'[\phi(z)]| |\phi'(z)| &= \frac{1}{v_{\log}[\phi(z)]} \mu(z) |\phi'(z)| v_{\log}[\phi(z)] |f'[\phi(z)]| \\ &\leq \frac{1}{L_1} \|\phi\|_\mu \|f\|_{\log}. \end{aligned}$$

Now suppose that $j \geq 2$. Then we have that

$$\begin{aligned} \mu(z) |f'[\phi(z)]| |\phi'(z)| &\leq \|f\|_{\log} \frac{\mu(z)}{v_{\log}[\phi(z)]} |\phi'(z)| \\ &\leq \|f\|_{\log} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} \frac{\frac{\|F_j\|_{\log}}{\log(j+1)}}{h_j(|\phi(z)|)}. \end{aligned}$$

By hypothesis, Lemma 2.2 (with $t = |\phi(z)| \in A_j^\phi$), and Corollary 2.4, it follows that there is an $L_2 > 0$, depending on neither f nor z in this case $j \geq 2$, such that

$$(14) \quad \mu(z) |f'[\phi(z)]| |\phi'(z)| \leq L_2 \|f\|_{\log}.$$

Since the respective estimates (13) and (14) hold when $j = 1$ and $j \geq 2$, it follows that there is an \tilde{L} such that Inequality (12) holds for arbitrary $z \in \mathbb{D}$.

Finally, since each $f \in \mathcal{B}^\mu$ is analytic on \mathbb{D} , we have that for all such f ,

$$f[\phi(0)] = f(0) + \int_0^{\phi(0)} f'(s) ds = f(0) + \int_0^{\phi(0)} \frac{1}{v_{\log}(s)} v_{\log}(s) f'(s) ds,$$

from which it follows that

$$\begin{aligned} |f[\phi(0)]| &\leq |f(0)| + \int_0^{\phi(0)} \left| \frac{1}{v_{\log}(s)} \right| |v_{\log}(s)| |f'(s)| |ds| \\ &\leq |f(0)| + \int_0^{\phi(0)} \left| \frac{1}{v_{\log}(s)} \right| \|f\|_{\log} |ds| \\ &= |f(0)| + \int_0^{\phi(0)} \left| \frac{1}{v_{\log}(s)} \right| |ds| \|f\|_{\log}. \end{aligned}$$

Letting C denote the integral quantity in the above expression, which is finite, non-negative, and independent of f , we conclude that for all $f \in \mathcal{B}^\mu$,

$$|f[\phi(0)]| + \|f \circ \phi\|_\mu \leq |f(0)| + (C + \tilde{L}) \|f\|_{\log} \leq Q(|f(0)| + \|f\|_{\log}) = Q\|f\|_{\mathcal{B}^{\log}},$$

where $Q := \max\{1, C + \tilde{L}\}$. The converse portion of the theorem follows, thus completing the proof of the theorem. \blacksquare

4. AUXILIARY RESULTS ON COMPACTNESS

Now that we know which analytic self-maps ϕ of \mathbb{D} induce bounded composition operators from the log-Bloch space to the μ -Bloch space, we now turn to the issue of compactness, which we handle by studying the essential norm of C_ϕ . Theorem 5.1, stated and proved in the section that follows, gives an expression that is equivalent to the essential norm of C_ϕ in this setting. We will need to prove two auxiliary facts first, the first of which is the following lemma that appears in [26] and also in [27]. In both of these places, we point out a typographical error (“point evaluation functionals on X ” there, as one can see from Relation (16) in [27], should be “point evaluations on Y ”).

Lemma 4.1 (Tjani). *Let X, Y be two Banach spaces of analytic functions on \mathbb{D} . Suppose that*

- (1) *Point evaluation functionals on Y are bounded.*
- (2) *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*
- (3) *$T : X \rightarrow Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then T is a compact operator if and only if given a bounded sequence $(f_j)_{j \in \mathbb{N}}$ in X such that $f_j \rightarrow 0$ uniformly on compact sets, $(Tf_j)_{j \in \mathbb{N}} \rightarrow 0$ in Y .

We will use Tjani’s Lemma to prove the following fact, whose purpose is to prove the lemma that follows it. Though it is familiar to some readers, we will sketch the details of the proof to maintain completeness.

Lemma 4.2. *Suppose that μ is a weight on \mathbb{D} . Then the following statements hold:*

- (A) *For each compact $K \subset \mathbb{D}$, there is a $C_K \geq 0$ such that for all $f \in \mathcal{B}^\mu$ and $z \in K$, we have that*

$$|f(z)| \leq C_K \|f\|_{\mathcal{B}^\mu}.$$

- (B) *Every point evaluation functional on \mathcal{B}^μ is bounded.*

(C) The closed unit ball of \mathcal{B}^μ is compact in the topology of uniform convergence on compacta.

(D) If γ is a weight on \mathbb{D} and $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic, with $C_\phi : \mathcal{B}^\mu \rightarrow \mathcal{B}^\gamma$, then C_ϕ is continuous with respect to the compact-open subspace topologies on \mathcal{B}^μ and \mathcal{B}^γ .

Proof. (A): Suppose that $K \subset \mathbb{D}$ is compact, and let μ be a weight on \mathbb{D} . Then there is an $r \in [0, 1)$ such that $|z| \leq r$ for all $z \in K$, and for each z , the line segment $[0, z] \subset \overline{D}(0, r)$, the compact Euclidean disk centered at 0 with radius r . Since μ is a weight, it follows that there must be a $Q > 0$ such that $\mu(s) \geq Q$ for all $s \in D(0, r)$, and in particular, for all s on the line segment $[0, z]$, for all $z \in K$. Thus for all $z \in K$ and $s \in [0, z]$, we have that $1/\mu(s) \leq 1/Q$. which implies that for all $f \in \mathcal{B}^\mu$ and $z \in K$,

$$|f(z)| \leq |f(0)| + \int_0^z \frac{\|f\|_\mu}{\mu(s)} |ds| \leq \left(1 + \frac{1}{Q}\right) \|f\|_\mu \leq \left(1 + \frac{1}{Q}\right) \|f\|_{\mathcal{B}^\mu}.$$

Thus the proof of (A) is complete.

(B): Part (B) follows immediately from Part (A).

(C): It follows from Part (A) that the unit ball of \mathcal{B}^μ is uniformly bounded on compacta. Therefore, by Montel's theorem (cf. [6]), any sequence $(f_n)_{n \in \mathbb{N}}$ in this unit ball must be a normal family, and there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ that must converge uniformly on compacta to some $f \in \mathcal{H}(\mathbb{D})$. By another of Montel's Theorems, $f_{n_k} \rightarrow f'$ uniformly on compacta as well and pointwise in particular. We then have that for any $z \in \mathbb{D}$,

$$\mu(z)|f'(z)| = \lim_{k \rightarrow \infty} \mu(z)|f'_{n_k}(z)| \leq 1,$$

since $\|f_{n_k}\|_\mu \leq \|f_{n_k}\|_{\mathcal{B}^\mu} \leq 1$ for each $k \in \mathbb{N}$. Thus $f \in \mathcal{B}_\mu$, and we thus have shown that the closed unit ball of \mathcal{B}_μ is compact in the compact-open topology, as desired.

(D): Let $f_n \rightarrow f$ uniformly in \mathcal{B}^μ on compacta, so that in turn, $f_n \circ \phi \rightarrow f \circ \phi$ uniformly on compacta as well. By assumption, $C_\phi(f) \in \mathcal{B}^\gamma$ and $C_\phi(f_n) \in \mathcal{B}^\gamma$ for all $n \in \mathbb{N}$, and the statement of Part (D) follows. \square

Combining the above lemma and Lemma 4.1, we obtain the following principal auxiliary result of this section:

Lemma 4.3. *Let μ_1 and μ_2 be weights on \mathbb{D} , and suppose that $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic. Then $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ is compact if and only if given a bounded sequence $(f_j)_{j \in \mathbb{N}}$ in \mathcal{B}^{μ_1} such that $f_j \rightarrow 0$ uniformly on compact subsets of \mathbb{D} , then $\|C_\phi(f_j)\|_{\mathcal{B}^{\mu_2}} \rightarrow 0$ as $j \rightarrow \infty$.*

5. THE ESSENTIAL NORM AND HENCE COMPACTNESS OF C_ϕ FROM \mathcal{B}^{\log} TO \mathcal{B}^μ

5.1. An expression that is equivalent to the essential norm. The main theorem of this section is the following equivalence result, which we prove after stating two preliminary lemmas below it.

Theorem 5.1.

$$(15) \quad \|C_\phi\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} \sim \limsup_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}},$$

if ϕ is a holomorphic self-map of \mathbb{D} and μ is a weight on \mathbb{D} such that C_ϕ is bounded between \mathcal{B}^{\log} and \mathcal{B}^μ .

5.2. Two auxiliary facts. In order to prove the above theorem, we will need the following two lemmas, which we will prove for the sake of completeness. For the first lemma, we will require the following notation: For $r \in [0, 1]$, define the linear *dilation operator* $K_r : \mathcal{H}(\mathbb{D}) \rightarrow \mathcal{H}(\mathbb{D})$ by $K_r f = f_r$, where f_r , for each $f \in \mathcal{H}(\mathbb{D})$, is given by $f_r(z) = f(rz)$. For more information about this operator, see [9].

Lemma 5.2. *Let $r \in [0, 1]$. Then the following statements hold:*

(A) \mathcal{B}^{\log} is a K_r -invariant subspace of $\mathcal{H}(\mathbb{D})$; moreover, we have that

$$\|K_r\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^{\log}} \leq 1.$$

(B) If $r \neq 1$, then K_r is compact on \mathcal{B}^{\log} .

Proof:

(A): We omit the proof of this part of the lemma, since it can be obtained by combining [22, Thm. 1, Part (e)] in the case $\beta = \alpha = 1$ there with Theorem 6.2 in the present paper.

(B): To prove this part of the Lemma, we can invoke Lemma 4.1, provided we can show that for any bounded sequence $\{f_j\}$ in \mathcal{B}^{\log} such that $f_j \rightarrow 0$ uniformly on compacta,

$$\|K_r f_j\|_{\mathcal{B}^{\log}} = |K_r f_j(0)| + \sup_{z \in \mathbb{D}} v_{\log}(z) |(K_r f_j)'(z)| \rightarrow 0$$

as $j \rightarrow \infty$. Indeed, we observe that the first term on the right side of the above equation tends to zero as $j \rightarrow \infty$, since $K_r f_j(0) = f_j(0)$ and f_j converges to 0 on compacta in \mathbb{D} , by assumption. Furthermore, we have that $f_j' \rightarrow 0$ uniformly on compacta (cf. [6, p. 142-151]). This fact and boundedness of v_{\log} together imply that

$$\sup_{z \in \mathbb{D}} v_{\log}(z) |(K_r f_j)'(z)| = r \sup_{z \in \mathbb{D}} v_{\log}(z) |f_j'(rz)| \leq r \|v_{\log}\|_{\infty} \sup_{z \in \mathbb{D}} |f_j'(rz)| \rightarrow 0$$

as $j \rightarrow \infty$. The statement in Part (B) of the Lemma follows.

Lemma 5.3. *Suppose that $(t_j)_{j \in \mathbb{W}}$ is an increasing sequence in $[0, 1)$ that converges to 1, and let $f \in \mathcal{H}(\mathbb{D})$. Then we have that $(t_j[f']_{t_j})_{j \in \mathbb{W}}$ converges uniformly to f' on compact subsets of \mathbb{D} .*

Proof: For completeness, we supply some of the details of the proof: Let G be a compact subset of \mathbb{D} , and let $\varepsilon > 0$. Then $G \subset \overline{D}(0, r)$, where $\overline{D}(0, r)$ denotes the Euclidean disk with center at the origin in \mathbb{C} and radius $r \in [0, 1)$. Since $f \in \mathcal{H}(\mathbb{D})$, it follows that f' is uniformly continuous on $\overline{D}(0, r)$. One verifies, therefore, that $|f' - (f')_{t_j}| \rightarrow 0$ uniformly on $\overline{D}(0, r)$. Furthermore,

$$|(f')_{t_j} - t_j(f')_{t_j}| = (1 - t_j)|f'|_{t_j} \leq (1 - t_j)M \rightarrow 0 \text{ as } j \rightarrow \infty,$$

so the proof of the lemma can be completed by a straightforward put-and-take, followed by an $\varepsilon/2$ -argument involving the triangle inequality.

5.3. Proof of the main essential norm result. We are now prepared to complete the proof of Theorem 5.1:

Proof: Suppose that $\phi : \mathbb{D} \rightarrow \mathbb{D}$, and assume that ϕ is holomorphic. If ϕ is the zero function, then the statement of the theorem holds trivially. Therefore, we can

assume throughout the sequel that ϕ is not the zero function. Since $\|\cdot\|_{\mathcal{B}^\mu}$ is a norm, it follows that $\|\phi\|_{\mathcal{B}^\mu} > 0$. Let μ be a weight on \mathbb{D} . We set

$$E := \limsup_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}},$$

which is a finite, non-negative real number, by Theorem 3.1 and the fact that the monomial functions $F_j \in \mathcal{B}^{\log}$ for all $j \in \mathbb{N}$. Let $K : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$ be linear and also compact, and define the normalized monomial function sequence $(f_j)_{j \in \mathbb{N}}$ in \mathcal{B}^{\log} by

$$f_j := \frac{F_j}{\|F_j\|_{\log}}.$$

We note that

$$(16) \quad f_j \rightarrow 0 \text{ uniformly on compacta in } \mathbb{D} \text{ as } j \rightarrow \infty.$$

Since the reverse triangle inequality holds for seminorms, we have that

$$\begin{aligned} \|C_\phi f_j\|_\mu - \|K f_j\|_\mu &\leq \|C_\phi f_j - K f_j\|_\mu \\ &\leq \|C_\phi f_j - K f_j\|_{\mathcal{B}^\mu} \\ &\leq \|C_\phi - K\| \|f_j\|_{\mathcal{B}^{\log}} \\ &= \|C_\phi - K\| \|f_j\|_{\log} \\ &= \|C_\phi - K\|. \end{aligned}$$

Combining this estimate and the equations

$$\|C_\phi f_j\|_\mu = \left\| C_\phi \left(\frac{F_j}{\|F_j\|_{\log}} \right) \right\|_\mu = \frac{1}{\|F_j\|_{\log}} \|C_\phi F_j\|_\mu = \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}},$$

we obtain the following inequality:

$$\frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} - \|K f_j\|_\mu \leq \|C_\phi - K\|.$$

By taking the limsup of both sides of the above inequality as $j \rightarrow \infty$ and using Relation (16) above along with Lemma 4.3, we can conclude that

$$\|C_\phi - K\| \geq E.$$

Therefore, we have that

$$(17) \quad \|C_\phi\|_e^{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu} \geq E.$$

Inequality (17) implies that the proof of the theorem will be complete if we can show that the left hand side of the inequality is bounded above by the product of a constant and E . First, we record the following fact for use later in the proof: Corollary 2.4 implies that we can, in particular, find an $N \in \mathbb{N}$ such that

$$(18) \quad \frac{\|F_m\|_{\log}}{\log(m+1)} < \frac{3}{2e} \text{ for all } m \in \mathbb{N} \text{ such that } m \geq N.$$

As we noted after we defined it in Equation (2), the sequence $(r_j)_{j \in \mathbb{W}}$ satisfies $r_j \in [0, 1)$ for all $j \in \mathbb{W}$, so Lemma 5.2, Part (B) implies that

$$(19) \quad \text{For all } j \in \mathbb{W} \text{ } K_{r_j} \text{ is compact on } \mathcal{B}^{\log}.$$

This fact and [7, p. 178, Prop. 3.5] together imply that

$$(20) \quad C_\phi K_{r_j} : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu \text{ is compact for all } j \in \mathbb{W}.$$

By Lemma 5.2, Part (A), we have that

$$(21) \quad \|K_{r_j}\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^{\log}} \leq 1 \text{ for all } j \in \mathbb{W}.$$

Also, for all $j \in \mathbb{W}$, we have that

$$(22) \quad \begin{aligned} \|C_\phi\|_{\mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu(\mathbb{D})} &= \inf \{ \|C_\phi - K\| \text{ such that } K : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu \text{ is compact} \} \\ &\leq \|C_\phi - K_j\| \\ &= \sup_{\{f \in \mathcal{B}^{\log} : \|f\|_{\mathcal{B}^{\log}} \leq 1\}} \|(C_\phi - C_\phi K_{r_j})f\|_{\mathcal{B}^\mu}. \end{aligned}$$

Thus the proof will be complete if we can show that the norm inside the above supremum is bounded above by the product of a constant that does not depend on the choice of $f \in \mathcal{B}^{\log}$ and E . We will break up the norm inside the supremum above into three pieces, each of which we will show is bounded above by a constant times E . Suppose for the moment that $f \in \mathcal{B}^{\log}$ and that $\|f\|_{\mathcal{B}^{\log}} \leq 1$. Since $r_j \rightarrow 1^-$ as $j \rightarrow \infty$ and each f_j is continuous, then for any $\varepsilon > 0$, we can choose $N' \in \mathbb{W}$ such that

$$(23) \quad \text{for all } j \in \mathbb{W} \text{ such that } j \geq N', \quad |f[\phi(0)] - f[r_j\phi(0)]| < E.$$

Since $r_j \in [0, 1)$ for all $j \in \mathbb{W}$ and $(r_j)_{j \in \mathbb{W}}$ is increasing with limit 1, we have by Lemma 5.3 that $(r_j[f']_{r_j})_{j \in \mathbb{W}}$ converges to 0 uniformly on compacta in \mathbb{D} . Since $\{w \in \mathbb{D} : |w| \leq r_l\}$ is compact, $(r_j[f']_{r_j})_{j \in \mathbb{W}}$ converges to 0 uniformly on $\{w \in \mathbb{D} : |w| \leq r_l\}$ for each $l \in \mathbb{W}$, and $\|\phi\|_{\mathcal{B}^\mu} > 0$, then for all $l \in \mathbb{W}$, we can find an $N_l \in \mathbb{W}$ such that for all $j \in \mathbb{W}$ with $j \geq N_l$ and all $z \in \mathbb{D}$ such that $|\phi(z)| \leq r_l$, we have

$$|r_j(f')_{r_j}[\phi(z)] - f'[\phi(z)]| < \frac{E}{\|\phi\|_{\mathcal{B}^\mu}}.$$

Thus for all $l \in \mathbb{W}$ and for all $j \in \mathbb{W}$ such that $j \geq N_l$, we have that

$$\sup_{z \in \mathbb{D} : |\phi(z)| \leq r_l} \mu(z) |r_j(f')_{r_j}[\phi(z)] - f'[\phi(z)]| |\phi'(z)| < \|\phi\|_\mu \frac{E}{\|\phi\|_{\mathcal{B}^\mu}} \leq E.$$

We record the above statement more briefly below for use later:

$$(24) \quad \sup_{z \in \mathbb{D} : |\phi(z)| \leq r_l} \mu(z) |r_j(f')_{r_j}[\phi(z)] - f'[\phi(z)]| |\phi'(z)| < E \text{ for all } j \in \mathbb{W} \text{ such that } j \geq N_l.$$

On the other hand, we have that

$$(25) \quad \text{for all } l, j \in \mathbb{W}, \quad \sup_{|\phi(z)| > r_l} \mu(z) |f'[\phi(z)] - r_j f'[r_j\phi(z)]| |\phi'(z)| \leq s_l(1) + s_l(r_j),$$

where for $\rho \in [0, 1]$, the expression $s_l(\rho)$ is given by

$$s_l(\rho) = \sup_{|\phi(z)| > r_l} \mu(z) |f'[\rho\phi(z)]| |\phi'(z)|,$$

if this quantity is finite for all $\rho \in [0, 1]$. Indeed, this quantity is finite for all such ρ , as we will now prove.

Assume now that $\rho \in [0, 1]$ and estimate $s_l(\rho)$ in this case. For such values of ρ and all $l \in \mathbb{W}$ such that $l \geq N$, we deduce that $s_l(\rho)$ is no larger than

$$\begin{aligned}
& \sup_{z \in \bigcup_{m=l+1}^{\infty} A_m^{\phi}} \mu(z) |f'[\rho\phi(z)]| |\phi'(z)| \\
& \leq \sup_{z \in \bigcup_{m=l}^{\infty} A_m^{\phi}} \mu(z) |f'[\rho\phi(z)]| |\phi'(z)| \\
& \leq \sup_{m \geq l} \sup_{z \in A_m^{\phi}} \mu(z) |f'[\rho\phi(z)]| |\phi'(z)| \frac{m |\phi(z)|^{m-1} \log(m+1) v_{\log}[\rho\phi(z)] \|F_m\|_{\log}}{m |\phi(z)|^{m-1} \log(m+1) v_{\log}[\rho\phi(z)] \|F_m\|_{\log}} \\
& < \frac{3}{2e} \sup_{m \geq l} \sup_{z \in A_m^{\phi}} \mu(z) |f'[\rho\phi(z)]| |\phi'(z)| \frac{m |\phi(z)|^{m-1} \log(m+1) v_{\log}[\rho\phi(z)]}{m |\phi(z)|^{m-1} v_{\log}[\rho\phi(z)] \|F_m\|_{\log}} \\
& = \frac{3}{2e} \sup_{m \geq l} \sup_{z \in A_m^{\phi}} |f'[\rho\phi(z)]| \mu(z) |m[\phi(z)]^{m-1} \phi'(z)| \frac{\log(m+1) v_{\log}[\rho\phi(z)]}{m |\phi(z)|^{m-1} v_{\log}[\rho\phi(z)] \|F_m\|_{\log}} \\
& = \frac{3}{2e} \sup_{m \geq l} \sup_{z \in A_m^{\phi}} |f'[\rho\phi(z)]| v_{\log}[\rho\phi(z)] \mu(z) |(\phi^m)'(z)| \frac{\log(m+1)}{m |\phi(z)|^{m-1} v_{\log}[\rho\phi(z)] \|F_m\|_{\log}} \\
& \leq \frac{3 \|f\|_{\mathcal{B}^{\log}}}{2e} \sup_{m \geq l} \frac{\|\phi^m\|_{\mu}}{\|F_m\|_{\log}} \sup_{z \in A_m^{\phi}} \frac{\log(m+1)}{m |\phi(z)|^{m-1} v_{\log}[\rho\phi(z)]} \\
& \leq \frac{3}{2e} \sup_{m \geq l} \frac{\|\phi^m\|_{\mu}}{\|F_m\|_{\log}} \sup_{z \in A_m^{\phi}} \frac{\log(m+1)}{m |\phi(z)|^{m-1} v_{\log}[\rho\phi(z)]} \\
& \leq \frac{3}{2e} \sup_{m \geq l} \frac{\|\phi^m\|_{\mu}}{\|F_m\|_{\log}} \sup_{z \in A_m^{\phi}} \frac{\log(m+1)}{m |\phi(z)|^{m-1} v_{\log}[\phi(z)]} \\
& = \frac{3}{2e} \sup_{m \geq l} \frac{\|\phi^m\|_{\mu}}{\|F_m\|_{\log}} \sup_{z \in A_m^{\phi}} \frac{1}{h_m[\phi(z)]} \\
& < \left(\frac{2e^L}{L}\right) \left(\frac{3}{2e}\right) \sup_{m \geq l} \frac{\|\phi^m\|_{\mu}}{\|F_m\|_{\log}} \\
& = \frac{3e^{L-1}}{L} \sup_{m \geq l} \frac{\|\phi^m\|_{\mu}}{\|F_m\|_{\log}} \\
& < \infty.
\end{aligned}$$

In the above chain of relations, the third inequality follows from Inequality (18). By definition of $\|\cdot\|_{\mathcal{B}^{\log}}$, the fourth inequality above is obtained, and the fifth inequality holds by the assumption that $\|f\|_{\mathcal{B}^{\log}} \leq 1$. The sixth inequality above follows from the fact that the continuous extension to $[0, 1]$ of μ_3 is increasing on $[0, 1]$, and the seventh inequality is a consequence of Lemma 2.2. Hence, s_l for $l \geq N$ is a well-defined, real-valued function on $[0, 1]$, as claimed, and we now have one of three estimates that are needed to complete the proof of the theorem.

Note in particular by separate estimation of $s_l(1)$ and $s_l(r_j)$, which are bounded by the second-from-the-bottom quantity in the above large chain of inequalities, that for sufficiently large l and any $j \in \mathbb{W}$,

$$\begin{aligned}
s_l(1) + s_l(r_j) & \leq \frac{3e^{L-1}}{L} \sup_{m \geq l} \frac{\|\phi^m\|_{\mu}}{\|F_m\|_{\log}} + \frac{3e^{L-1}}{L} \sup_{m \geq l} \frac{\|\phi^m\|_{\mu}}{\|F_m\|_{\log}} \\
(26) \qquad \qquad & = \frac{6e^{L-1}}{L} \sup_{m \geq l} \frac{\|\phi^m\|_{\mu}}{\|F_m\|_{\log}}.
\end{aligned}$$

Now, for all $j, l \in \mathbb{W}$, we have that $\|(C_\phi - C_\phi K_{r_j})f\|_{\mathcal{B}^\mu}$

$$\begin{aligned}
&= \|(f - f_{r_j}) \circ \phi\|_{\mathcal{B}^\mu} \\
&= |f[\phi(0)] - f[r_j\phi(0)]| + \sup_{z \in \mathbb{D}} \mu(z) |f'[\phi(z)] - r_j f'[r_j\phi(z)]| |\phi'(z)| \\
(27) \quad &\leq |f[\phi(0)] - f[r_j\phi(0)]| + E_{j,1}^l(f) + E_{j,2}^l(f),
\end{aligned}$$

where

$$E_{j,1}^l(f) := \sup_{|\phi(z)| \leq r_l} \mu(z) |f'[\phi(z)] - r_j f'[r_j\phi(z)]| |\phi'(z)|$$

and

$$E_{j,2}^l(f) := \sup_{|\phi(z)| > r_l} \mu(z) |f'[\phi(z)] - r_j f'[r_j\phi(z)]| |\phi'(z)|.$$

Quantity (27) above can be rewritten as

$$(28) \quad |f[\phi(0)] - f[r_j\phi(0)]| + E_{j,1}^l(f) + s_l(1) + s_l(r_j).$$

Now let $l \in \mathbb{W}$ satisfy $l \geq N$, and let $j \in \mathbb{W}$ satisfy $j \geq N'_l := \max\{N_l, N'\}$.

Indeed, by Inequality (27), which is bounded above by Quantity (28), we have that

$$\begin{aligned}
\|(C_\phi - C_\phi K_{r_j})f\|_{\mathcal{B}^\mu} &\leq |f[\phi(0)] - f[r_j\phi(0)]| + E_{j,1}^l(f) + E_{j,2}^l(f) \\
&= |f[\phi(0)] - f[r_j\phi(0)]| + E_{j,1}^l(f) + s_l(1) + s_l(r_j) \\
(29) \quad &< E + E_{j,1}^l(f) + s_l(1) + s_l(r_j) \\
(30) \quad &< E + E + s_l(1) + s_l(r_j) \\
(31) \quad &\leq E + E + \frac{6e^{L-1}}{L} \sup_{m \geq l} \frac{\|\phi^m\|_\mu}{\|F_m\|_{\log}} \\
&= 2E + \frac{6e^{L-1}}{L} \sup_{m \geq l} \frac{\|\phi^m\|_\mu}{\|F_m\|_{\log}}.
\end{aligned}$$

Inequality (29) follows from Statement (23), and Inequality (30) is due to Statement (24). Inequality (31) follows from Equation (26), together with the inequality that precedes it. Since the chain of inequalities above holds for all $l \in \mathbb{W}$ such that $l \geq N$ and in turn for the j 's in \mathbb{W} such that $j \geq N'_l$, we can conclude, by Equation (22) and the above estimates, that the essential norm of C_ϕ is bounded above by

$$\frac{2L + 6e^{L-1}}{L} E.$$

This completes the proof of the theorem ■

5.4. A characterization of the symbols generating compact composition operators. Theorem 5.1 and Corollary 2.4 together immediately imply the following characterization of analytic symbols ϕ that generate compact $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$, thus extending results in [28, 33].

Theorem 5.4. *Let μ be a weight on \mathbb{D} . Suppose that $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic. Then $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\mu$ is compact if and only if either of the following equations holds:*

$$\begin{aligned}\lim_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\log(j+1)} &= 0. \\ \lim_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} &= 0.\end{aligned}$$

5.5. Essential norms of C_ϕ from the log-Bloch to α -Bloch spaces. The following result establishing essential norm equivalences and characterizing compact C_ϕ from the log-Bloch space to α -Bloch spaces is a direct consequence of Theorem 5.1:

Corollary 5.5. *Let $\phi : \mathbb{D} \rightarrow \mathbb{D}$ be analytic, and suppose that $\alpha \geq 0$. Then the following statements hold.*

- (1) *The essential norm of the continuous operator $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^\alpha$ satisfies*

$$\|C_\phi\|_e \sim \limsup_{j \rightarrow \infty} \frac{\|\phi^j\|_\alpha}{\|F_j\|_{\log}}.$$

In particular, this operator is compact if and only if

$$\lim_{j \rightarrow \infty} \frac{\|\phi^j\|_\alpha}{\|F_j\|_{\log}} = 0.$$

- (2) *The essential norm of the continuous operator $C_\phi : \mathcal{B}^{\log} \rightarrow \mathcal{B}^{\log}$ satisfies*

$$\|C_\phi\|_e \sim \limsup_{j \rightarrow \infty} \frac{\|\phi^j\|_{\log}}{\|F_j\|_{\log}}.$$

In particular, this operator is compact if and only if

$$\lim_{j \rightarrow \infty} \frac{\|\phi^j\|_{\log}}{\|F_j\|_{\log}} = 0.$$

6. A PAIRWISE NORM-EQUIVALENT FAMILY OF GENERALIZED LOG BLOCH SPACES

In this section, we observe that our results concerning composition operators can be extended to a more general family of spaces that include the log Bloch space. In addition, this general family of logarithmic Bloch-type spaces can be defined on the unit ball \mathbb{B}_n of \mathbb{C}^n induced by the Euclidean inner product, and we will point out that these spaces are pairwise norm-equivalent to each other and the log Bloch space for all $n \in \mathbb{N}$. If μ is a continuous positive function on \mathbb{B}_n , then the μ -Bloch space $\mathcal{B}^\mu(\mathbb{B}_n)$ is defined to be the Banach space of holomorphic functions on f on \mathbb{B}_n such that

$$b_\mu^f := \sup_{z \in \mathbb{B}_n} \mu(z) |(\nabla f)(z)|$$

is finite, the Banach space structure arising from the norm given by

$$\|f\|_{\mathcal{B}^\mu(\mathbb{B}_n)} := |f(0)| + b_\mu^f,$$

which we denote more briefly by $\|f\|_{\mathcal{B}^\mu}$ as in the case $n = 1$.

Suppose that $\theta > 1$ and that $k = 1$ or 2 . Then we define the (k, θ) -log Bloch space $\mathcal{B}_{k, \theta}^{\log}(\mathbb{B}_n)$ to be $\mathcal{B}^\mu(\mathbb{B}_n)$, where the weight $\mu := v_\theta^{(k)}$, and $v_\theta^{(k)}$ is in turn given

by $v_\theta^{(k)}(z) = (1 - |z|^k) \log \frac{\theta}{1 - |z|^k}$. For the sake of brevity, we denote the norm of $f \in \mathcal{B}_{k,\theta}^{\log}(\mathbb{B}_n)$ here by $\|f\|_{k,\theta}$. Also, consistent with standard notation in the case $n = 1$, we adopt the notation $\mathcal{B}^{\log}(\mathbb{B}_n) := \mathcal{B}_{1,3}^{\log}(\mathbb{B}_n)$ and call this space the *log Bloch space* of \mathbb{B}_n .

The goal of this section is to prove a norm equivalence result from which we will be able to extend, in the section that follows, the main results of this paper to the (k, θ) -log Bloch spaces defined above, with the more stringent condition $\theta \geq e$, although we will prove the above-mentioned pairwise norm-equivalence among these spaces on \mathbb{B}_n for all $n \in \mathbb{N}$, not just on \mathbb{D} . We leave open the question of whether analogues of our main results on composition operators hold in the case of \mathbb{B}_n ; moreover, after our results were obtained, S. Stević pointed out to us the papers [12], [22], [23], [25], and [24], in which composition operators (and some natural generalizations of them) are considered on spaces formed by replacing the “ $(1 - |z|)$ ” and the logarithm in the definition of the norm on $\mathcal{B}_{k,\theta}^{\log}(\mathbb{B}_n)$ in the special case $k = 1$ by various respective powers of these quantities. The present paper can be viewed as complementing, in some ways, a number of these results by Stević and his collaborators.

To prove the pairwise norm-equivalence of the (k, θ) -logarithmic Bloch spaces for suitable (k, θ) , we first need the following auxiliary fact:

Lemma 6.1. *Suppose that $\mu : (0, 1) \rightarrow (0, \infty)$ is increasing, and assume that for all $t \in (0, 1)$, we have that*

$$(32) \quad \mu\left(\frac{1}{2}t\right) \geq \frac{1}{2}\mu(t).$$

Then $\mu(1 - t) \sim \mu(1 - t^2)$.

Proof: By the following inequality for all $t \in [0, 1]$,

$$\frac{1}{2}(1 - t^2) \leq 1 - t \leq 1 - t^2,$$

Inequality (32), and the assumption that μ is increasing, we have that for these t 's,

$$\frac{1}{2}\mu(1 - t^2) \leq \mu\left(\frac{1}{2}(1 - t^2)\right) \leq \mu(1 - t) \leq \mu(1 - t^2).$$

■

We are now prepared to prove the following norm equivalence result, which shows that our main results for composition operators on the log Bloch space extend to the (k, θ) -log Bloch spaces defined earlier in this section, for suitable k 's and θ 's. We know of no reference containing the proof, which may be known to some readers, so we provide a sketch of the details for the convenience of other readers. For $\theta > 1$, we will make use of $\mu_\theta : (0, 1] \rightarrow [0, \infty)$ defined by $\mu_\theta(t) = t \log(\theta/t)$.

Theorem 6.2. *Let $\alpha, \beta \geq e$. Then we have that*

$$\begin{aligned} v_\alpha^{(2)}(z) &\sim v_\beta^{(2)}(z) \sim v_\beta^{(1)}(z), \\ \mathcal{B}_{2,\alpha}^{\log} &= \mathcal{B}_{2,\beta}^{\log} = \mathcal{B}_{1,\beta}^{\log}, \\ \|f\|_{2,\alpha} &\sim \|f\|_{2,\beta} \sim \|f\|_{1,\beta}, \end{aligned}$$

as f varies through these coinciding spaces.

Proof: Let $\alpha, \beta \geq e$. The second and third set of equivalences above immediately follow from the top set of equivalences, which we now prove. We can assume with no loss of generality, that $\alpha \leq \beta$. We prove the leftmost of these two equivalences first. The following two inequalities respectively follow from the facts that (i) \log is increasing and (ii) for all $\theta \geq e$, μ_θ is increasing:

$$(33) \quad v_\alpha^{(2)}(z) \leq v_\beta^{(2)}(z) \leq \log \beta \quad \text{for all } z \in \mathbb{B}_n.$$

Since we have by L'Hopital's Rule that $v_\beta^{(2)}(z)/v_\alpha^{(2)}(z) \rightarrow 1$ as $|z| \rightarrow 1^-$, one checks that there is a $\delta > 0$ such that for all $z \in \mathbb{B}_n$ with $\delta < |z| < 1$,

$$(34) \quad v_\beta^{(2)}(z) \leq \frac{3}{2} v_\alpha^{(2)}(z).$$

One verifies that for $\theta > 1$, μ_θ is bounded with a removable discontinuity on its graph, at the origin, and the restriction of μ_θ to $(1 - \delta^2, 1)$ for any $\delta \in (0, 1)$, is therefore bounded away from 0; in particular, there is an $m_{\alpha, \delta} > 0$, depending only on δ and α , such that $v_\alpha^{(2)}(z) \geq m_{\alpha, \delta}$ for all $z \in \overline{\delta \mathbb{B}_n}$. By the rightmost inequality in Relation (33), it follows that for these z 's ,

$$v_\beta^{(2)}(z) \leq \log \beta = \log \beta \frac{m_{\alpha, \delta}}{m_{\alpha, \delta}} \leq \log \beta \frac{v_\alpha^{(2)}(z)}{m_{\alpha, \delta}} = \frac{\log \beta}{m_{\alpha, \delta}} v_\alpha^{(2)}(z).$$

We then obtain the leftmost equivalence in the top collection of relations in the conclusion of the theorem from this fact, the leftmost inequality in Relation (33), and the fact that Inequality (34) holds for all $z \in \mathbb{B}_n$ such that $\delta < |z| < 1$.

We now prove the second equivalence in the top relation appearing in the conclusion of the theorem. One first checks that since $\beta \geq e$, μ_β is increasing and concave. In particular, μ_β satisfies the hypotheses of Lemma 6.1, which allows us to deduce that $\mu_\beta(1 - t) \sim \mu_\beta(1 - t^2)$. It follows that the second equivalence in the top relation in the conclusion of the theorem holds. \blacksquare

7. BOUNDEDNESS, COMPACTNESS, AND ESSENTIAL NORMS OF COMPOSITION OPERATORS FROM $\mathcal{B}_{k, \theta}^{\log}(\mathbb{D})$ TO $\mathcal{B}^\mu(\mathbb{D})$

Theorem 6.2 and the theorems presented in Section 3 and 5 together immediately imply the following more general results on composition operators from log-Bloch type spaces to weighted Bloch spaces of \mathbb{D} . To date, various members of this family have been ambiguously given the same name, “logarithmic Bloch space”. We do not claim to have found the entire collection of $(k, \theta) \in \mathbb{R}^2$ to which the results of this paper extend.

Theorem 7.1. *Suppose that $\theta \geq e$, and assume that $k = 1$ or 2 . Suppose that $\phi : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic, and assume that μ is a weight on \mathbb{D} . Then the following statements hold:*

(A) $C_\phi : \mathcal{B}_{k, \theta}^{\log} \rightarrow \mathcal{B}^\mu$ is bounded if and only if

$$\sup_{j \in \mathbb{N}} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} < \infty.$$

if and only if

$$\sup_{j \in \mathbb{N}} \frac{\|\phi^j\|_\mu}{\log(j+1)} < \infty.$$

(B) $C_\phi : \mathcal{B}_{k,\theta}^{\log} \rightarrow \mathcal{B}^\mu$ is compact if and only if

$$\lim_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} = 0.$$

if and only if

$$\lim_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\log(j+1)} = 0.$$

(C) If $C_\phi : \mathcal{B}_{k,\theta}^{\log} \rightarrow \mathcal{B}^\mu$, then

$$\|C_\phi\|_e^{\mathcal{B}_{k,\theta}^{\log} \rightarrow \mathcal{B}^\mu} \sim \limsup_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\|F_j\|_{\log}} \sim \limsup_{j \rightarrow \infty} \frac{\|\phi^j\|_\mu}{\log(j+1)}.$$

8. ACKNOWLEDGMENTS

The authors are grateful to Stevo Stević for kindly emailing the papers [12], [22], [23], [24], and [25] and also for kindly making many very helpful comments on the manuscript.

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